LAGUERRE COLLOCATION APPROACH FOR CONTINUOUS HYBRID BLOCK METHOD FOR SOLVING INITIAL VALUE PROBLEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Supported by tetfund TERTIARY EDUCATION TRUST FUND

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Received: December 08, 2016 **Accepted:** February 27, 2017

Abstract: In this paper, an implicit one-step method with three off-grid points for numerical solution of second order initial value problems of ordinary differential equation has been developed by collocation and interpolation technique. The one-step method was developed using Laguerre polynomial as basis function and the method was augmented by the introduction of off-step points in order to bring zero stability and upgrade the order of consistency of the method. An advantage of the derived continuous scheme is that it can produce several outputs of solution at the off-grid points without requiring additional interpolation. Numerical examples are solved with the aid of MAPLE software package. We observed that the results obtained from the method converged faster when the numbers of off-step points were increased and this validate the consistency and zero stability of the method.

Keywords: Collocation, continuous scheme, interpolation, Laguerre polynomial

Introduction

The general second order Initial Valve Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form:

$y'' = f(x, y, y'), y'(a) = z_o, y(a) = y_o, x\epsilon[a.b]$ (1)

where f is continuous in [a,b], is often encountered in areas such as satellite tracking/warning systems, celestial mechanics, mass action kinetics, solar systems and molecular biology Aladeselu (2007). Many of such problems may not be easily solved analytically; hence numerical schemes are developed to solve (1).

Some researchers have attempted the solution of (1) using LMMs without reduction to system of first order ODEs (Lambert, 1991; Kayode, 2005; Adesanya *et al.*, 2009; Yahaya and Badmus, 2009). Awoyemi (1999) proposed a continuous scheme based on collocation which was found to have better error estimate and provided approximation at all interior points of the interval of consideration. The main setback of the scheme proposed by Awoyemi (1999) is in the need to develop computer sub-programs needed to initialize the starting values; hence, much time is lost and the cost of implementation is high. In view of these disadvantages, many researchers concentrated efforts on advancing the numerical solution of IVPs in ODEs. One of the outcomes is the development of a class of methods called block method. The method, which shall briefly be discussed in the next section simultaneously generates approximate at different grid points in the interval of integration and is less expensive in terms of the number of function evaluations compared to Runge-Kutta methods.

Block Methods

Block methods are formulated in terms of LMMs. They provide the traditional advantage of one-step methods of being self-starting and permit easy change of step length (Lambert, 1973). Another important feature of the block approach is that all the discrete schemes are of uniform order and are obtained from a single continuous formula in contrast to the non-self starting predictor-corrector approach. In what now immediately follows, we shall develop the new method with Laguerre polynomial as basis function.

Development of the Method

In this section, we intend to derive a continuous representation of a one-step method which will be used to generate the main method and other methods required to set up the block method. We set out by approximating the analysis solution of problem (1) with a Laguerre polynomial of the form:

$Y(x) = \sum_{j=o}^{k} a_j\, L_j(x) = y(x)$ (2)

where

$L_{j+1}(x) = (-1)^{n+1} e^{x} \frac{d^{n+1}}{dx^{n+1}}(e^{-x}x^{n+1})$

so that

$L_o(x) = 1, L_1(x) = (x-1), L_2(x) = x^2 - 4x + 2, L_3(x) = x^3 + 9x^2 + 18x - 6.$

on the partition

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_n = b$$

on the integration interval [a, b], with a constant step size h, given by

$$h = x_{n+1} - x_n;\, n = 0,1, \ldots n-1.$$

We need to interpolate at two points to be able to approximate (1) and, to make this happen, we proceed by arbitrarily selecting an off-step point, x_{n+v},v∈(0,1) in (x_n, x_{n+1}) in such a manner that the zero-stability of the main method is guaranteed. Then (2) is interpolated at $x_{n+i}, i = 0, v$ and its second derivative is collocated at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ so as to obtain a system of seven equations each of degree six i.e. $k = 6$

$\sum_{j=o}^{6} a_j\, L_j(x) = y(x)$ (3)

$\sum_{j=0}^{6} a_j L_j''(x) = f(x, y, y')$ (4)

Let us arbitrarily set $v = \frac{1}{2}$ then collocating (4) at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and interpolating (3) at $x_{n+i}, i = 0, \frac{1}{2}$ lead to system of equations of the form;

$$f_n = 2a_2 + a_3(6x_n - 18) + a_4(12x_n^2 - 96x_n + 144) + a_5(20x_n^3 - 300x_n^2 + 1800x_n + 1800) + a_6(30_n^4 - 720x_n^3 + 5400x_n^2 - 14400x_n + 10800)$$

$$f_{n+1/4} = 2a_2 + a_3(6x_{n+1/4} - 18) + a_4(12x_{n+1/4}^2 - 96x_{n+1/4} + 144) + a_5(20x_{n+1/4}^3 - 300x_{n+1/4}^2 + 1800x_{n+1/4} + 1800) + a_6(30_{n+1/4}^4 - 720x_{n+1/4}^3 + 5400x_{n+1/4}^2 - 14400x_{n+1/4} + 10800)$$

$$f_{n+1/2} = 2a_2 + a_3(6x_{n+1/2} - 18) + a_4(12x_{n+1/2}^2 - 96x_{n+1/2} + 144) + a_5(20x_{n+1/2}^3 - 300x_{n+1/2}^2 + 1800x_{n+1/2} + 1800) + a_6(30_{n+1/2}^4 - 720x_{n+1/2}^3 + 5400x_{n+1/2}^2 - 14400x_{n+1/2} + 10800)$$

$$f_{n+3/4} = 2a_2 + a_3(6x_{n+3/4} - 18) + a_4(12x_{n+3/4}^2 - 96x_{n+3/4} + 144) + a_5(20x_{n+3/4}^3 - 300x_{n+3/4}^2 + 1800x_{n+3/4} + 1800) + a_6(30_{n+3/4}^4 - 720x_{n+3/4}^3 + 5400x_{n+3/4}^2 - 14400x_{n+3/4} + 10800)$$

$$f_{n+1} = 2a_2 + a_3(6x_{n+1} - 18) + a_4(12x_{n+1}^2 - 96x_{n+1} + 144) + a_5(20x_{n+1}^3 - 300x_{n+1}^2 + 1800x_{n+1} + 1800) + a_6(30_{n+1}^4 - 720x_{n+1}^3 + 5400x_{n+1}^2 - 14400x_{n+1} + 10800)$$

$$y_n = a_0 + a_1(x_n - 1) + a_2(x_n^2 - 4x_n + 2) + a_3(x_n^3 - 9x_n^2 + 18x_n - 6) + a_4(x_n^4 - 16x_n^3 + 72x_n^2 - 96x_n - 24) + a_5(x_n^5 - 25x_n^4 + 300x_n^3 - 900x_n^2 + 600x_n - 120) + a_6(x_n^6 - 36x_n^5 + 450x_n^4 - 2400x_n^3 + 5400x_n^2 - 4320x_n + 720)$$

$$y_{n+1/2} = a_0 + a_1(x_{n+1/2} - 1) + a_2(x_{n+1/2}^2 - 4x_{n+1/2} + 2) + a_3(x_{n+1/2}^3 - 9x_{n+1/2}^2 + 18x_{n+1/2} - 6) + a_4(x_{n+1/2}^4 - 16x_{n+1/2}^3 + 72x_{n+1/2}^2 - 96x_{n+1/2} - 24) + a_5(x_{n+1/2}^5 - 25x_{n+1/2}^4 + 300x_{n+1/2}^3 - 900x_{n+1/2}^2 + 600x_{n+1/2} - 120) + a_6(x_{n+1/2}^6 - 36x_{n+1/2}^5 + 450x_{n+1/2}^4 - 2400x_{n+1/2}^3 + 5400x_{n+1/2}^2 - 4320x_{n+1/2} + 720)$$

We solve the system of seven equations by MAPLE to obtain the value of the unknown parameters $a_j, j = 0(1)6$.

Substituting $a_j{}'s$ into (2) yields a continuous implicit hybrid one-step method in the form:

$$Y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2[\textstyle\sum_{j=0}^{1}\beta_j(x)f_{n+j} + \beta_{\frac{1}{4}}(x)\,f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}] \quad (5)$$

where $\alpha_j(x) and\ \beta_j(x)$ are continuous coefficient $y_{n+j} = y(x_n + jh)$ is the numerical approximation of the analytical solution at $x_{n+j} and\ f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.

Equation (5) yields the α_j and β_j as the following continuous function of t:

$$\alpha_o = -1,\ \ \alpha_{1/2} = 2,\ \ \beta_{1/4} = \frac{h^2}{15},\ \ \beta_{1/2} = \frac{13h^2}{120}, \beta_{3/4} = \frac{h^2}{15}, \beta_1 = \frac{h^2}{240} \quad (6)$$

Evaluating (5) at x_{n+1}, the main method is obtained as:

$$y_{n+1} + y_n - 2y_{n+1/2} = \frac{h^2}{240}\left[f_n + 16f_{n+\frac{1}{4}} + 26f_{n+\frac{1}{2}} + 16f_{n+\frac{3}{4}} + f_{n+1}\right] \quad (7)$$

To derive the block method, additional equations are needed since equation (7) alone will not be sufficient for the solution. The additional equations can be obtained by evaluating the first derivative of equation (5):

$$Y'(x) = \frac{1}{h}\left[\alpha'_0(x)y_n + \alpha'_{\frac{1}{2}}(x)y_{n+\frac{1}{2}}\right] + h(\textstyle\sum_{j=0}^{1}\beta'_j(x)f_{n+j} + \beta'_{\frac{1}{4}}(x)f_{n+\frac{1}{4}} + \beta'_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta'_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}) \quad (8)$$

at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ respectively, This yields the following discrete derivative schemes:

$$720hy'_n + 1440y_n - 1440y_{n+1/2} = h^2[-53f_n - 144f_{n+\frac{1}{4}} + 30f_{n+\frac{1}{2}} - 16f_{n+\frac{3}{4}} + 3f_{n+1}] \quad (9)$$

$$2880hy'_{n+1/4} + 5760y_n - 5760y_{n+1/2} = h^2[39f_n + 70f_{n+\frac{1}{4}} - 144f_{n+\frac{1}{2}} + 42f_{n+\frac{3}{4}} - 7f_{n+1}] \quad (10)$$

$$720hy'_{n+1/2} + 1440y_n - 1440y_{n+1/2} = h^2[5f_n + 104f_{n+\frac{1}{4}} + 78f_{n+\frac{1}{2}} - 8f_{n+\frac{3}{4}} + f_{n+1}] \quad (11)$$

$$2880hy'_{n+3/4} + 5760y_n - 5760y_{n+1/2} = h^2[31f_n + 342f_{n+\frac{1}{4}} + 768f_{n+\frac{1}{2}} + 314f_{n+\frac{3}{4}} - 15f_{n+1}] \quad (12)$$

$$720hy'_{n+1} + 1440y_n - 1440y_{n+1/2} = h^2[3f_n + 112f_{n+\frac{1}{4}} + 126f_{n+\frac{1}{2}} + 240f_{n+\frac{3}{4}} + 59f_{n+1}] \quad (13)$$

Equations (7), (9), (10), (11), (12) and (13) are solved simultaneously to obtain the following explicit results:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{90}\left[7f_n + 24f_{n+\frac{1}{4}} + 6f_{n+\frac{1}{2}} + 8f_{n+\frac{3}{4}}\right] \quad (14)$$

$$y_{n+1/2} = y_n + \frac{1}{2}hy'_n + \frac{h^2}{1440}\left[53f_n + 144f_{n+\frac{1}{4}} - 30f_{n+\frac{1}{2}} + 16f_{n+\frac{3}{4}} - 3f_{n+1}\right] \quad (15)$$

$$y'_{n+1} = y'_n + \frac{h}{90}\left[7f_n + 32f_{n+\frac{1}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{4}} + 7f_{n+1}\right] \quad (16)$$

$$y'_{n+1/2} = y'_n + \frac{h}{360}\left[29f_n + 124f_{n+\frac{1}{4}} + 24f_{n+\frac{1}{2}} + 4f_{n+\frac{3}{4}} - f_{n+1}\right] \quad (17)$$

$$y'_{n+1/4} = y'_n + \frac{h}{2880}\left[251f_n + 646f_{n+\frac{1}{4}} - 264f_{n+\frac{1}{2}} + 106f_{n+\frac{3}{4}} - 19f_{n+1}\right] \quad (18)$$

$$y'_{n+3/4} = y'_n + \frac{h}{320}\left[27f_n + 102f_{n+\frac{1}{4}} + 72f_{n+\frac{1}{2}} + 42f_{n+\frac{3}{4}} - 3f_{n+1}\right] \quad (19)$$

Analysis of the Method

The basic properties of the derived Scheme are discussed.

The Explicit Scheme (14-19) derived are discrete Scheme belonging to the class of LMM of the form

$$\textstyle\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} \quad (20)$$

The Linear differential operator L defined by

$$L[y(x);h] = \textstyle\sum_{j=0}^{k}[\alpha_j y(x+jh) - h^2\beta_j y^n(x+jh)] \quad (21)$$

Expanding (21) by Taylor series, we have

$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^q(x)$

where

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$

$C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k$

$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k$

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$C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \cdots + k^p\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{p-2}\beta_2 + \cdots + k^{q-2}\beta_k),$

$q \geq 3$

Order and error constant

Definition 1: The LMM (20) is said to be order P if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant.

Table 1: Showing the orders and error constants of the method

Equation numbers	Order (P)	Error constants
(14)	5	3.10019841 x 10^{-6}
(15)	5	2.46484953 x 10^{-6}
(16)	5	3.52371964 x 10^{-6}
(17)	5	1.55009921 x 10^{-6}
(18)	5	1.77052330 x 10^{-6}
(19)	5	2.71267361 x10^{-6}

Consistency

Definition 2: The LMM (20) is said to be consistent if it is of order P≥ 1 and its first and second characteristic polynomial defined as $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$ where Z satisfies $(i) \sum_{j=0}^{k} \alpha_j = 0, (ii) \rho'(1) = 0, (iii)\, \rho''(1) = 2!\,\sigma(1)$ (Lambart, 1973).

The discrete Schemes derived are all of order than one and satisfy the condition (i)-(iii)

Zero Stability of the block method

The block method is defined by Fatunla (1988) as

$$Y_m = \sum_{i=0}^{k} A_i + h \sum_{i=0}^{k} B_i F_{m-i}$$

where $Y_m = [y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+r-1}]^T$

$F_m = [f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+r-1}]^T$

$A_i's$ and $B_i's$ are chosen r x r matrix coefficient and $m = 0,1,2 \ldots$ represents the block number, $n = mr$, the first step number in the m-th block and r is the proposed block size.

The block method is said to be zero stable if the roots of $R_j, j = 1(1)k$ of the first characteristics polynomial is

$$\rho(R) = \det\left[\sum_{i=0}^{k} A_i R^{k-1}\right] = 0, A_0 = I$$

satisfies$|\mathbf{R_j}| \leq \mathbf{1}$, if one of the roots is +1, then the root is called Principal Root of $\rho(R)$.

Zero-stability for schemes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n+1} \\ y'_{n+1/2} \\ y'_{n+1/4} \\ y'_{n+3/4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n-1/4} \\ y'_{n-1/2} \\ y'_{n-1} \\ y'_{n} \end{bmatrix} + h \begin{bmatrix} \frac{32}{90} & \frac{12}{90} & \frac{32}{90} & \frac{7}{90} \\ \frac{124}{360} & \frac{24}{360} & \frac{4}{360} & \frac{-1}{360} \\ \frac{646}{2880} & \frac{-264}{2880} & \frac{106}{2880} & \frac{-19}{2880} \\ \frac{102}{320} & \frac{72}{320} & \frac{42}{320} & \frac{-3}{320} \end{bmatrix} \begin{bmatrix} f_{n+1/4} \\ f_{n+1/2} \\ f_{n+3/4} \\ f_{n+1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \frac{7}{90} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{27}{320} \end{bmatrix} \begin{bmatrix} f_{n-3/4} \\ f_{n-1/2} \\ f_{n-1/4} \\ f_n \end{bmatrix}$$

where

$$A^{(0)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ A^{(1)}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},\ B^{(0)}=\begin{bmatrix}\frac{32}{90}&\frac{12}{90}&\frac{32}{90}&\frac{7}{90}\\ \frac{124}{360}&\frac{24}{360}&\frac{4}{360}&\frac{-1}{360}\\ \frac{646}{2880}&\frac{-264}{2880}&\frac{106}{2880}&\frac{-19}{2880}\\ \frac{102}{320}&\frac{72}{320}&\frac{42}{320}&\frac{-3}{320}\end{bmatrix}\text{ and}$$

$$B^{(0)}=\begin{bmatrix}0&0&0&\frac{7}{90}\\0&0&0&\frac{29}{360}\\0&0&0&\frac{251}{2880}\\0&0&0&\frac{27}{320}\end{bmatrix}$$

The first characteristics polynomial of the scheme is

$\rho(\lambda)=\det[\lambda A^0 - A^1]$

$$\rho(\lambda)=det\left[\begin{pmatrix}\lambda&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&\lambda\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right]$$

$$\rho(\lambda)=det\begin{bmatrix}\lambda&0&0&-1\\0&\lambda&0&-1\\0&0&\lambda&-1\\0&0&0&\lambda-1\end{bmatrix}$$

$$\begin{vmatrix}\lambda&0&0&-1\\0&\lambda&0&-1\\0&0&\lambda&-1\\0&0&0&\lambda-1\end{vmatrix}=0$$

$\lambda^3(\lambda-1)=0$

$\lambda_1=\lambda_2=\lambda_3=0\ or\ \lambda_4=1$

We can see clearly that no root has modulus greater than one (i.e. $\lambda_i \le 1$) $\forall i$. The hybrid block method is zero stable.

Numerical Examples

Problem 1:

$y''=y', y(0)=0, y'(0)=-1, h=0.1$

Exact Solution: $y(x)=1-\exp(x)$

Problem 2:

$y''+1001y'+1000y=0, y(0)=1, y'(0)=-1, h=0.1$

Exact Solution: $y(x)=\exp(-x)$

Source: Adeniyi and Adeyefa (2013)

Table 2: The exact solution and the computed results from the proposed method for problem 1

x	Exact Solution	New Method	AA 2013	Error in New method	Error in [9]
0.1	-0.105170918	-0.1051709181	-0.105170902	9.999999E-11	0.160756E-07
0.2	-0.221402758	-0.2214027582	-0.221402723	0.245218E-09	0.351602E-07
0.3	-0.349858807	-0.3498588077	-0.34985857	0.734286E-09	0.237576E-06
0.4	-0.491824697	-0.4918246978	-0.491824433	0.835326E-09	0.2646413E-06
0.5	-0.64872127	-0.6487212709	-0.648720974	0.945324E-09	0.2967001E-06
0.6	-0.82211880	-0.8221188007	-0.822118466	0.734287E-09	0.3343905E-06
0.7	-1.013752707	-1.013752708	-1.013752329	0.193453E-08	0.3784705E-06
0.8	-1.225540928	-1.225540929	-1.225540498	0.156723E-08	0.4304925E-06
0.9	-1.459603111	-1.459603112	-1.45960262	0.165782E-08	0.4911569E-06
1.0	-1.718281828	-1.718281830	-1.718281267	0.224176E-08	0.561459E-06

Table 3: The exact solution and the computed results from the proposed method for problem 2

x	Exact Solution	New Method	AA 2013	Error in New method	Error in [8]
0.1	0.904837421	0.9048362113	0.90483742	1.20673E-06	0.23596E-09
0.2	0.818730753	0.8187465932	0.81873075	1.58401E-05	0.47798E-09
0.3	0.740818220	0.7408412156	0.74081822	2.29948E-05	0.58172E-09
0.4	0.670320046	0.6703188365	0.67032005	1.20963E-05	0.73564E-09
0.5	0.606530659	0.6065010742	0.60653066	2.99585E-05	0.81263E-09
0.6	0.548811633	0.5488584144	0.54881164	4.67783E-05	0.89405E-08
0.7	0.496585303	0.4965578820	0.49658630	2.74218E-05	0.99142E-08
0.8	0.449328964	0.4494673514	0.44932896	1.38387E-04	0.10172E-08
0.9	0.406569659	0.4067934283	0.40656966	2.22377E-04	0.10406E-07
1.0	0.367879441	0.3675984721	0.36787944	2.80969E-04	0.10714E-07

Conclusion

In this paper, it is observed from the tables that the results obtained from the method converged faster when the numbers of off-step points were increased. This validates the consistency and zero stability of the method.Generally, the performance of our method as noticed in **Table 2**, shows that the proposed method is more superior to the block method proposed by Adeniyi and Adeyefa (2013), for the same step sizes.However, even though the multiple finite difference method of Jator (2007) seemed to have produced a better result at most of the points of evaluation in **Table 3**, it should be noticed that the method had step number k=5 against our proposed method of step number k=1.

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